On the Appearance of the Basel Constant in Pattern Field Theory

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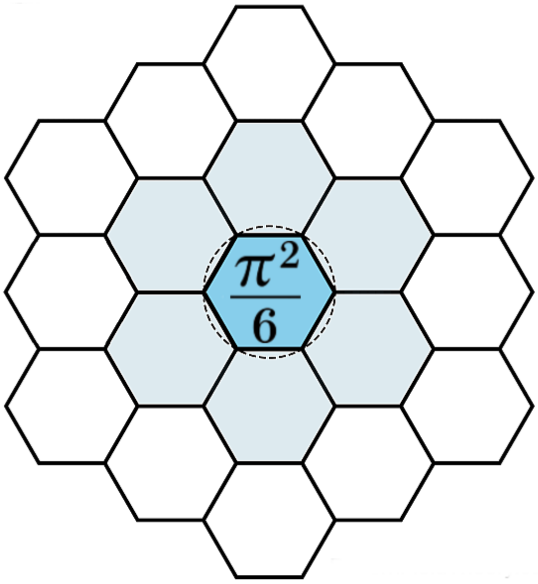


Abstract

The constant $\pi^2/6$, known as the Basel constant, arises in Pattern Field Theory (PFT) as a structural threshold associated with square-mode accumulation under closure constraints. This paper presents a rigorous derivation of $\zeta(2) = \pi^2/6$ and clarifies its interpretation within PFT. The derivation is classical and independent of PFT; the theory-specific contribution lies solely in how this invariant is interpreted as a saturation boundary for admissible configurations on the Allen Orbital Lattice. No modification of established mathematics is proposed.

1 The Basel Problem

The Basel problem concerns the exact value of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2),$$

where $\zeta(s)$ denotes the Riemann zeta function. Leonhard Euler first obtained the result $\zeta(2) = \pi^2/6$ in 1734. Modern rigor is provided by the theory of entire functions via Weierstrass factorization.

2 Canonical Product Representation of $\sin x$

The sine function is an entire function with simple zeros at $x = n\pi$ for all $n \in \mathbb{Z}$. A standard result is the canonical product

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right),$$

with convergence uniform on compact subsets of $\mathbb{C}$.

3 Series Expansion Near the Origin

The Taylor expansion of $\sin x/x$ about $x = 0$ is

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots,$$

so the coefficient of the x^2 term is $-1/6$.

4 Coefficient Comparison

Write the infinite product in the form

$$\prod_{n=1}^{\infty} (1 - a_n x^2), \quad a_n = \frac{1}{n^2\pi^2}.$$

Expanding to second order in x, only linear terms contribute:

$$\prod_{n=1}^{\infty} (1 - a_n x^2) = 1 - \left(\sum_{n=1}^{\infty} a_n\right) x^2 + O(x^4).$$

Thus the coefficient of x^2 is

$$-\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} = -\frac{1}{\pi^2}\zeta(2).$$

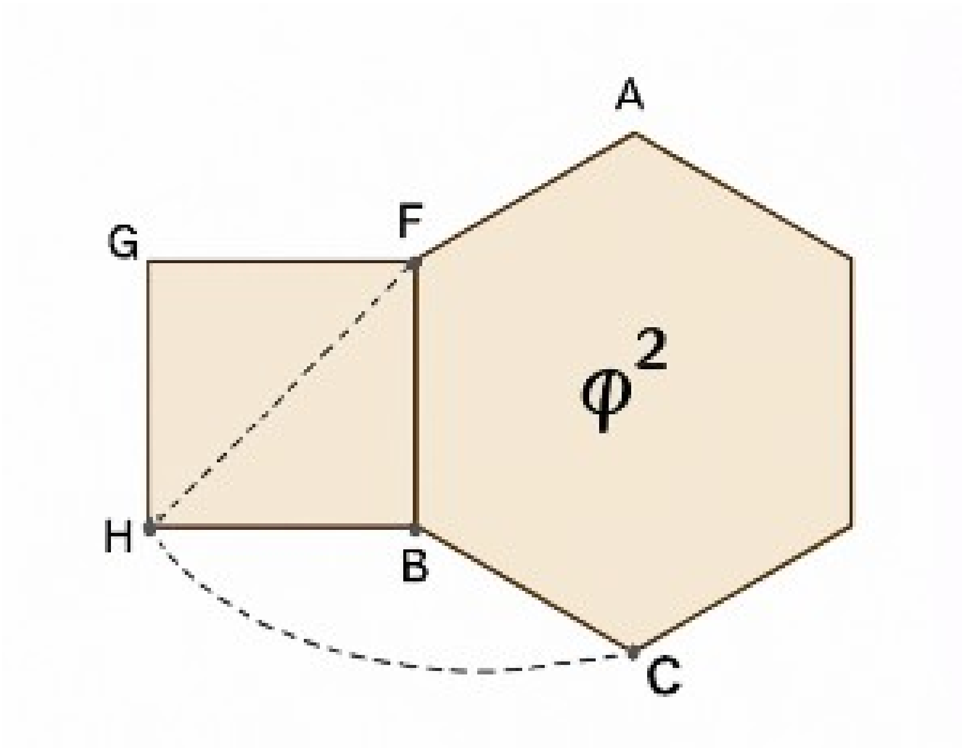
Equating this with the Taylor coefficient yields

$$-\frac{1}{\pi^2}\zeta(2) = -\frac{1}{6},$$

and therefore

$$\zeta(2) = \frac{\pi^2}{6}.$$

5 Interpretation Within Pattern Field Theory



Remark 1. *Independent structural constraints arising from discrete square-mode summation ($\pi^2/6$), planar hexagonal closure (6), and recursive stabilization (ϕ^2) converge on a common admissible equilibrium class within the Allen Orbital Lattice. The diagram indicates structural compatibility rather than derivation or causal dependence.*

The derivation above is purely mathematical and does not rely on Pattern Field Theory. Within PFT, however, the constant $\pi^2/6$ is interpreted as a *square-mode closure invariant.* It marks the saturation point at which discrete inverse-square contributions accumulate to a global rotational bound.

In this sense, $\pi^2/6$ does not introduce circular geometry by assumption. Rather, circular invariance emerges from discrete summation under admissibility and closure constraints on the Allen Orbital Lattice. This interpretation motivates the appearance of $\pi^2/6$ as a transition threshold in subsequent PFT constructions.

Glossary

Basel Constant The value $\pi^2/6$, equal to $\sum_{n=1}^{\infty} 1/n^2$.

Closure A structural condition requiring bounded accumulation of contributions.

Allen Orbital Lattice (AOL) The prime-indexed hexagonal substrate used in PFT.

Pattern Field Theory (PFT) A constraint-based framework in which physical observables emerge from admissible structural configurations.

Document Timestamp and Provenance

This document is part of Pattern Field Theory (PFT) and the Allen Orbital Lattice (AOL). It provides mathematical background used by subsequent papers in the series.